Generalized Harmonic Gauges and BSSN Evolutions of Single Black Holes

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Over the past few years, the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) 3+1 decomposition of Einstein's equations has achieved impressive results in the field of black hole simulations, producing stable runs of head-on collisions and binary systems up to one orbit. The effort to increase their lifetime and to extend this success to more general physical scenarios unfolds along multiple avenues, including improved excision algorithms, more sophisticated initial data and more suitable coordinate conditions.
In the latter setting, the recent success of black hole evolution by means of a generalized harmonic class of gauge conditions in the context of a harmonic decomposition of Einstein's equations [1] raises the question of whether the success of generalized harmonic gauges can be translated to codes based on the BSSN decomposition. The goal of this project is to investigate the distinctive properties and the numerical performance of such a class of gauge conditions for single black hole evolution in the framework of a BSSN-based code.

Introduction

Traditionally, numerical solutions of Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad (1)$$

have been carried out by means of 3+1 decompositions, i.e. by choosing a timelike direction in the spacetime and a corresponding foliation into three-dimensional, spacelike surfaces. One such 3+1 separation is known as the Arnowitt-Deser-Misner (ADM) decomposition, in which the line element takes on the general form:

$$ds^2 = -\alpha^2 dt^2 + h_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \qquad (2)$$

where α is referred to as the lapse, and the vector β^i as the shift. The fundamental evolved variables are the spatial metric h_{ij} and the extrinsic curvature K_{ij}, defined as:

$$K_{ij} = -h_i^k h_j^h \nabla_{(k} n_{h)} \qquad (3)$$

and $n_h = -\alpha \nabla_h t$.
From the numerical standpoint, the ADM equations have not been very successful. An alternative 3+1 decomposition, due to Baumgarte, Shapiro, Shibata and Nakamura, has been more successful and is now widely used. In this decomposition, the fundamental variables to evolve are the conformal spatial metric $\tilde{h}_{ij}$, the trace of the extrinsic curvature K, the traceless conformal curvature $\tilde{A}_{ij}$, the conformal factor Φ and the conformal connection functions $\tilde{\Gamma}^i$, related to the ADM variables by:

$$\begin{aligned} \Phi &= \frac{1}{12}\ln h && (4) \\ \tilde{h}_{ij} &= e^{-4\Phi} h_{ij} && (5) \\ \tilde{A}_{ij} &= e^{-4\Phi}(K_{ij} - \frac{1}{3} h_{ij} K) && (6) \\ \tilde{\Gamma}^i &= \tilde{h}^{jk} \tilde{\Gamma}^i_{jk} && (7) \end{aligned}$$

Since its introduction in [3], the BSSN decomposition of Einstein's equations has proved to possess the right features to ensure long-lived, constraint-preserving evolutions of black hole systems. Over the past five years, Penn State has developed MAYA, a BSSN-based code that uses a fixed mesh refinement system and an excision-type singularity handling routine. With MAYA, Sperhake et al. [4] were able to achieve stable runs of two head-on colliding black holes. With different singularity handling techniques, Brügmann, Tichy & Jansen [5] were also able to evolve a binary system through an entire orbit.
In the context of the MAYA code, a somewhat unsatisfactory feature is that the stability of head-on collision runs relies on the imposition of *a priori* trajectories for the two black holes. Such a gauge choice, on the other hand, is known to be unstable on long timescales; hence, it is necessary to switch to a different gauge after the merger in order to achieve long-term stability.
The recent successful application of generalized harmonic gauges in the framework of a harmonic decomposition of Einstein's equations brings about the question of whether an analogous choice of coordinates could also improve the stability of BSSN systems and simplify the gauge choice for black hole collisions.

Generalized Harmonic Gauges

A **generalized harmonic gauge** is a choice of coordinates x^μ that satisfy:

$$\Box x^\mu = H^\mu \qquad (8)$$

where the H^μ represent four freely specifiable functions of the spacetime coordinates.
The evolution equations for the lapse and the shift, expressed in terms of BSSN variables, read:

$$\begin{aligned} (\partial_t - \beta^i\partial_i)\alpha &= -\alpha^2 K - \alpha^2 n \cdot H \\ (\partial_t - \beta^i\partial_i)\beta^j &= -\alpha^2 e^{-4\Phi}\left[\tilde{h}^{jk}\partial_k \ln\alpha - (\tilde{\Gamma}^j + \tilde{h}^{jk}\partial_k\varphi) + \tilde{H}^j_\perp\right] \end{aligned}$$

with $n \cdot H := n_\mu H^\mu$ and $H^j_\perp := h^j_\mu H^\mu$.
Following [1] and [2], there are several options for evolving the source functions H^μ:

- The spatial projections $h^i_\mu H^\mu$ are set to zero, and $n_\mu H^\mu =: H_t$ is evolved via the following **wave equation**:

$$\Box H_t = \xi_1 \frac{\alpha - \alpha_0}{\alpha^\eta} + \xi_2 n^\mu \partial_\mu H_t \qquad (9)$$

  where ξ_1, ξ_2, η and α_0 are four positive constants. This particular source has the effect of driving the lapse towards α_0.
- An **algebraic condition** is imposed for all the H^μ by plugging the initial data into the equations for the lapse and the shift, solving for H^μ, and freezing their value to this form at all times.

We have found that the evolution of H^μ through a wave equation does not seem to lead to a stable scheme with MAYA. In the following section, we present our preliminary results obtained using algebraic conditions for the lapse and the shift.

Results

We tested the new gauge conditions in the single black hole scenario, and observed the resulting changes with respect to the benchmark run described in [6].
Both runs are carried out on a 35^3 grid, with four refinement levels and an excision mask of radius $r = 1$. The initial data is the spatial projection of a single black hole spacetime expressed in ingoing Eddington-Finkelstein (IEF) form:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{4M}{r} dt dr + (1 + \frac{2M}{r}) dr^2 + r^2 d\Omega^2 \qquad (10)$$

The stability of the runs can be assessed by monitoring the rate of change of the fundamental variables during evolution. In a static scenario, the rates are expected to remain small at all times.
The rate of change of α, K and Φ during evolution is shown in figures 1, 2 and 3, along with the corresponding result for the benchmark run.

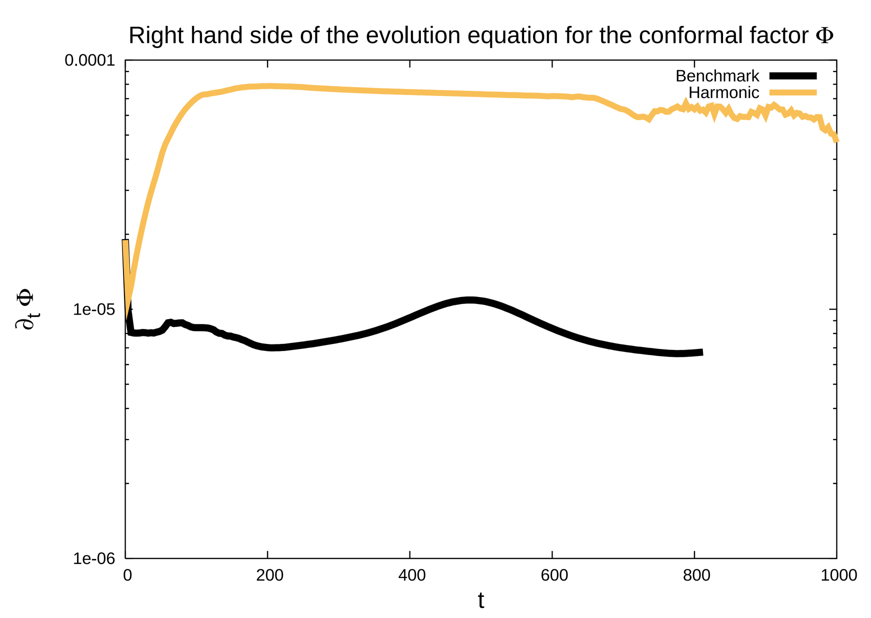

Figure 1: Right hand sides of the evolution equations for the conformal factor, for the benchmark single black hole run and for the generalized harmonic gauge run.

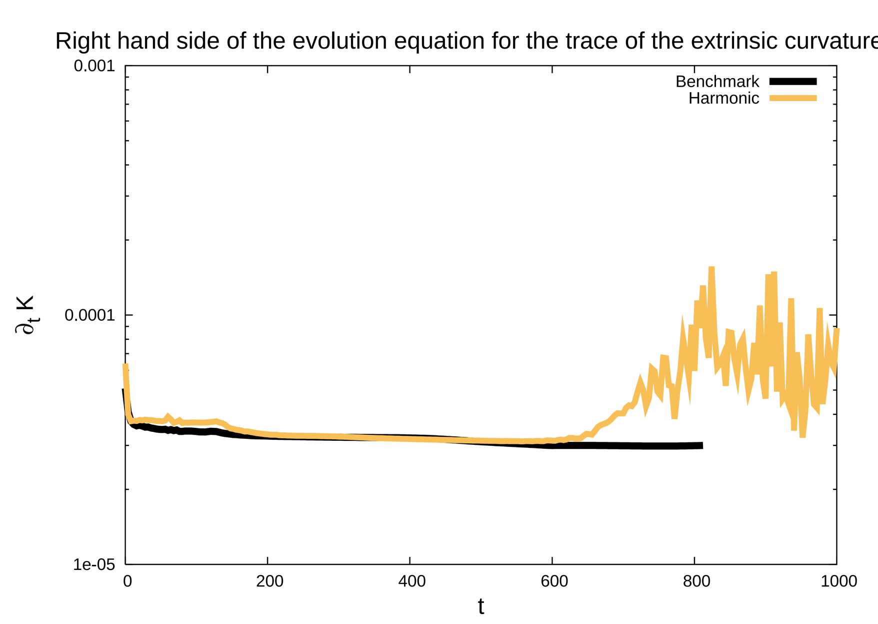

Figure 2: Right hand sides of the evolution equation for the trace of the extrinsic curvature, for the benchmark single black hole run and for the generalized harmonic gauge run.

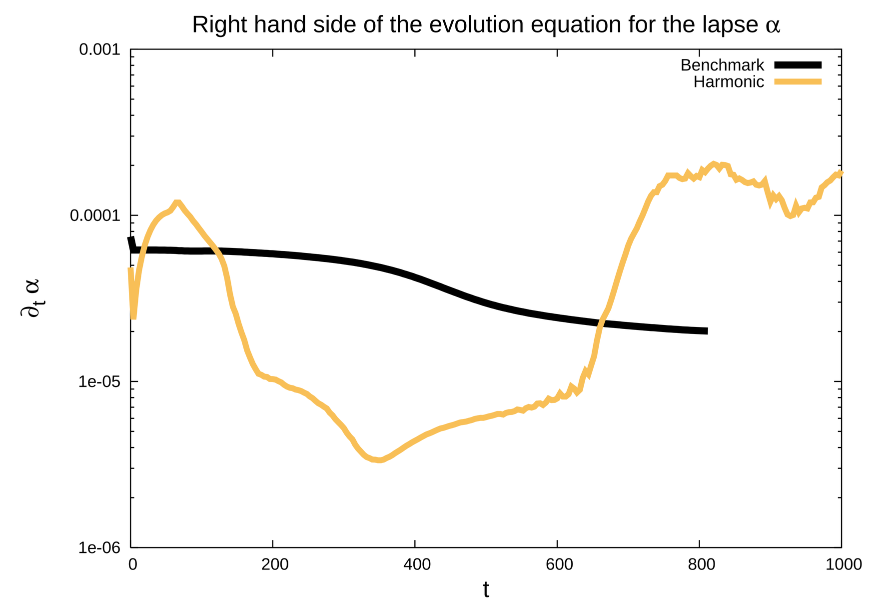

Figure 3: Right hand sides of the evolution equations for the lapse, for the benchmark single black hole run and for the generalized harmonic gauge run.

The data shows that the new gauge condition leads to a stable evolution for periods of the order of several hundreds times the mass of the black hole.

Conclusions

Harmonic gauge conditions have proved to be a promising alternative to standard slicing choices for black hole spacetimes, and have already outperformed previous results within the context of a harmonic decomposition.
We have been able to obtain a stable, single black hole evolution with a generalized harmonic gauge in the context of a BSSN-based code, and we are currently exploring ways to extend this result to more general physical scenarios.
The above results encourage further investigations: the vast set of choices for the source functions in equation (8) still remains mostly unexplored, and a proper classification of the possible choices for H^μ, along with their numerical suitability in different physical contexts, is necessary to tailor the slicing of the spacetime to the individual physical picture.

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