LSU
CENTER FOR COMPUTATION & TECHNOLOGY

5th Gulf Coast Gravity Meeting
Baton Rouge, April 18th 2009

Quasi-local angular momentum estimates via constant-expansion surfaces

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Outline

- MOTIVATION: estimating angular momentum in simulations of strongly gravitating systems;

- BACKGROUND:
  - -Angular momentum in General Relativity;
  - -Existing techniques for angular momentum estimates in Numerical Relativity;
  - -Constant-Expansion surfaces;

- RESULTS

- DISCUSSION, FUTURE WORK

Angular momentum in Relativity

- Newtonian concept:

$$\vec{J} = \vec{r} \wedge \vec{p}$$

conserved charge associated to rotations, through its conservation important properties of physical systems can be derived (stellar stability).

- Special Relativity:

$$J^{ab} = \epsilon^{ab}{}_{cd} r^c T^{d0} = \begin{bmatrix} 0 & xE & yE & zE \\ -xE & 0 & -J_z & J_y \\ -yE & J_z & 0 & -J_x \\ -zE & -J_y & J_x & 0 \end{bmatrix}$$

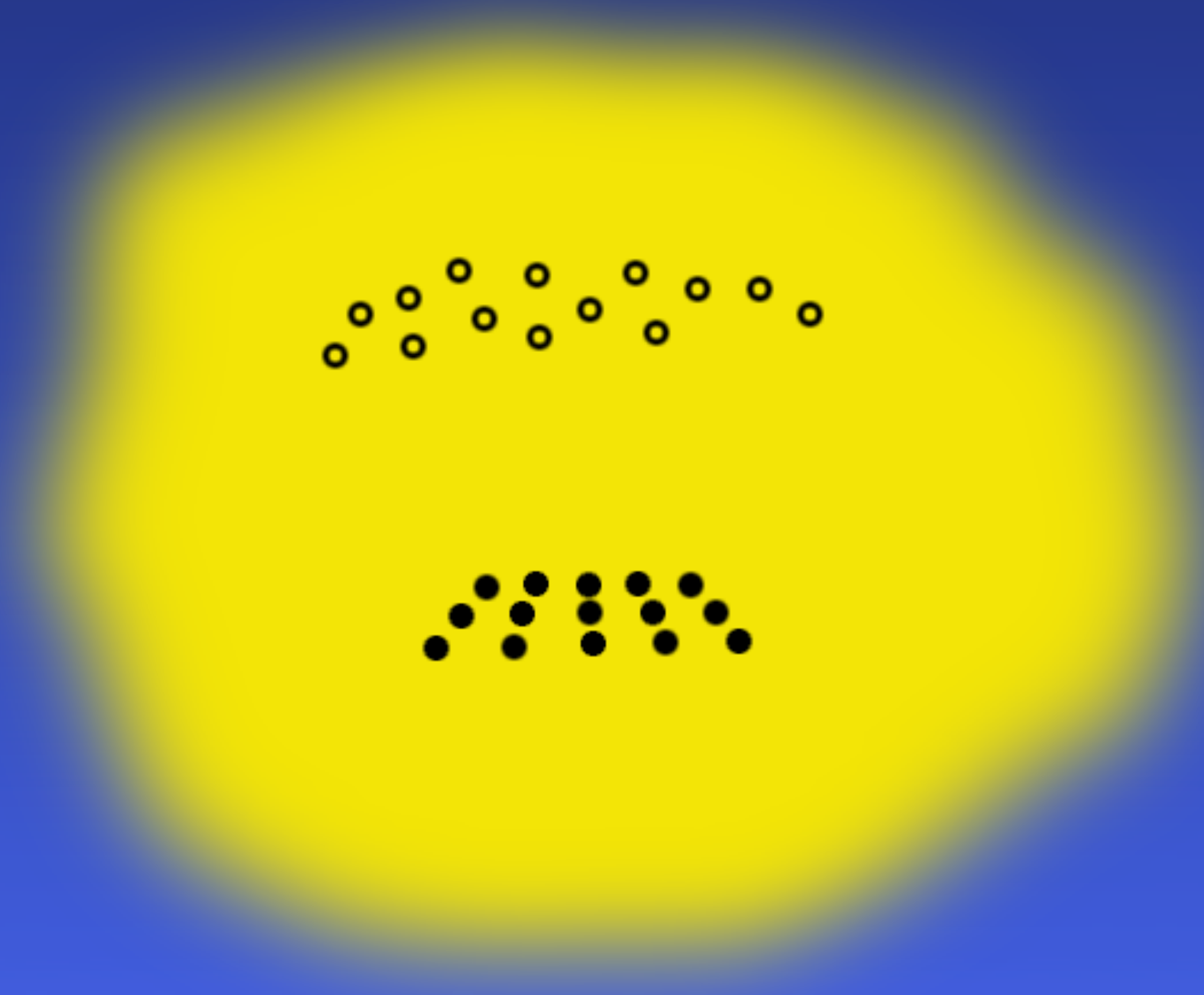
Angular momentum in Relativity

- General Relativity: apply to the gravitational field?

PROBLEMS	SOLUTIONS
LOCALIZATION: "local" energy-momentum and angular-momentum of the gravitational field are not observable (local flatness).	Global approach (spatial and null infinity)
	Quasi-local approach (extended but finite regions)
SYMMETRY & CONSERVATION: in a generic gravitational field, which symmetries? How to define traslations? Rotations?	Asymptotically flat spacetimes
	Symmetries of 2-surfaces
ORIGIN: Choose an appropriate origin with respect to which angular momentum is calculated (center of mass problem in GR). Important if one wants to compare different spacetimes!	Good cuts, nice sections, CE surfaces?

Angular momentum in numerical simulations

- In numerical simulations, however, it is not trivial to compute these quantities:
  - Only finite-distance surfaces are usually available - how can one calculate the limit to spatial or null infinity?
  - Gauge unknown, potentially non-trivial, not modifiable - how can one ensure that these corresponds to the correct chart/tetrad choice required by the global formalisms?



Angular momentum in numerical simulations

- Solutions proposed:

Approach	Solution
Common approach in numerical simulations	Use Bondi angular momentum, with chart and tetrad build from simulation's coordinates; extract on a set of constant coordinate radius surfaces and extrapolate to infinity
Gallo, Lehner & Moreschi; Deadman & Stewart	Use the transformation law to a Bondi chart/tetrad to rewrite the Bondi expressions at finite distance and in generic coordinates.
Our proposal	Use the Komar angular momentum on constant expansion surfaces, which select a radial coordinate, a tetrad and a center of mass.

Constant Expansion Surfaces

- In a 3+1 settimg, given a 2-surface, a null tetrad can be constructed from the normal to the spatial hypersurfaces, the normal to the 2-surface on the spatial hypersurfaces and two vectors tangent to the 2-surface, using:

$$\begin{aligned} \ell^a &= \frac{T^a + R^a}{\sqrt{2}} \\ n^a &= \frac{T^a - R^a}{\sqrt{2}} \\ m^a &= \frac{\theta^a + i\varphi^a}{\sqrt{2}} \end{aligned}$$

- The surface's outgoing null expansion is defined as:

$$\Theta_{(\ell)} = q^{ab}\nabla_a \ell_b$$

$$q^{ab} = g^{ab} + \ell^a n^b + n^a \ell^b$$

- A 2-surface is said to be a CE surface if the expansion of its outgoing null normal is constant across it.

- On simple, exact spacetimes, the expansion is asymptotically monotonic with the coordinate radius, and can be used as a radial coordinate;

Results

- Consider a Kerr spacetime in Kerr-Schild coordinates, and add a Lorentz boost and a traslation:

$$g_{ab} = \eta_{ab} + 2H k_a k_b$$

$$H = \frac{M r^3}{r^4 + a^2 z^2}$$

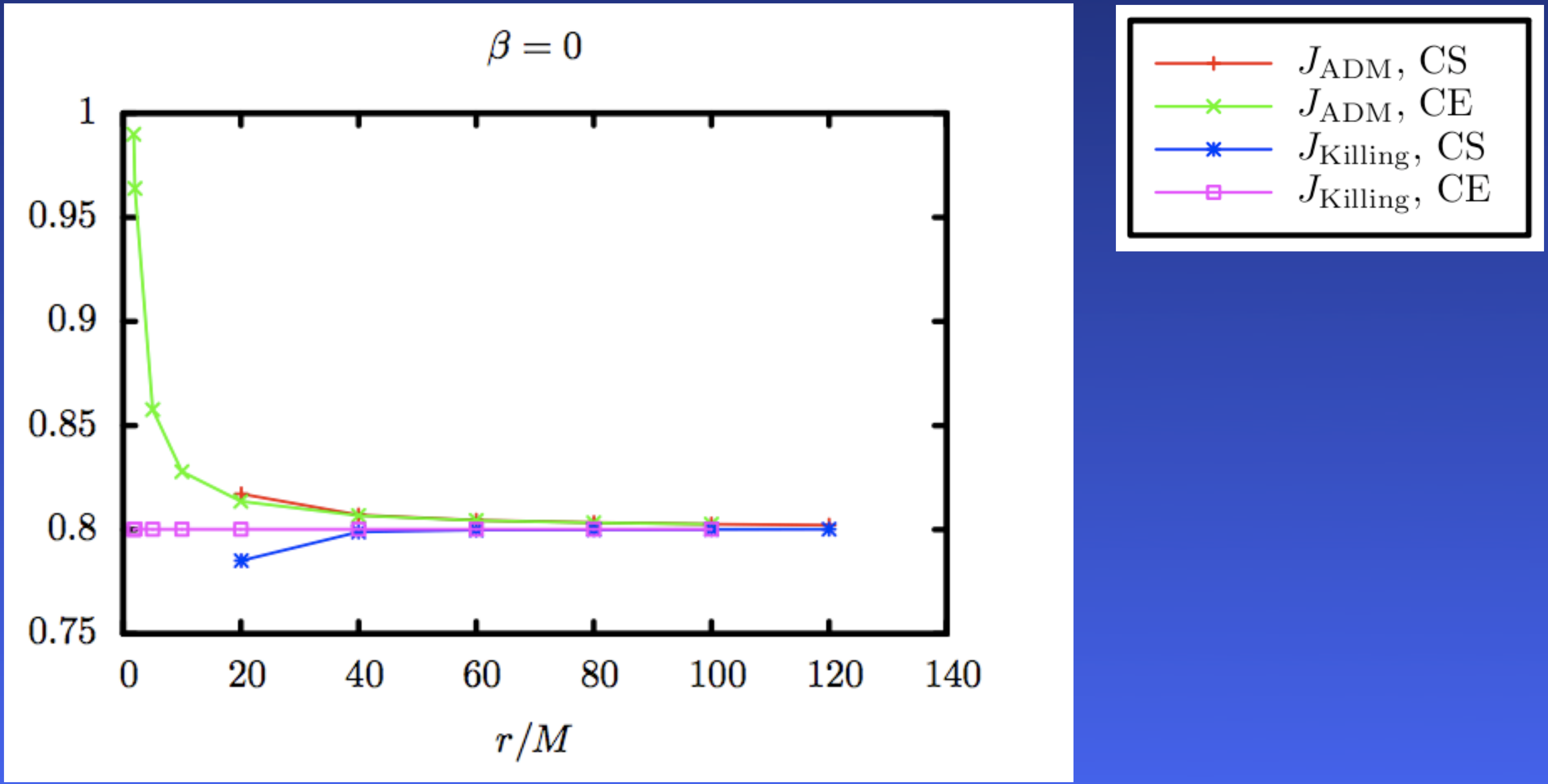
$$k_a dx^a = -\frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} - \frac{zdz}{r} - dt$$

$$\begin{aligned} t &\rightarrow \gamma(t - \beta y) \\ x &\rightarrow x + \Delta \\ y &\rightarrow \gamma(y - \beta t) \end{aligned}$$

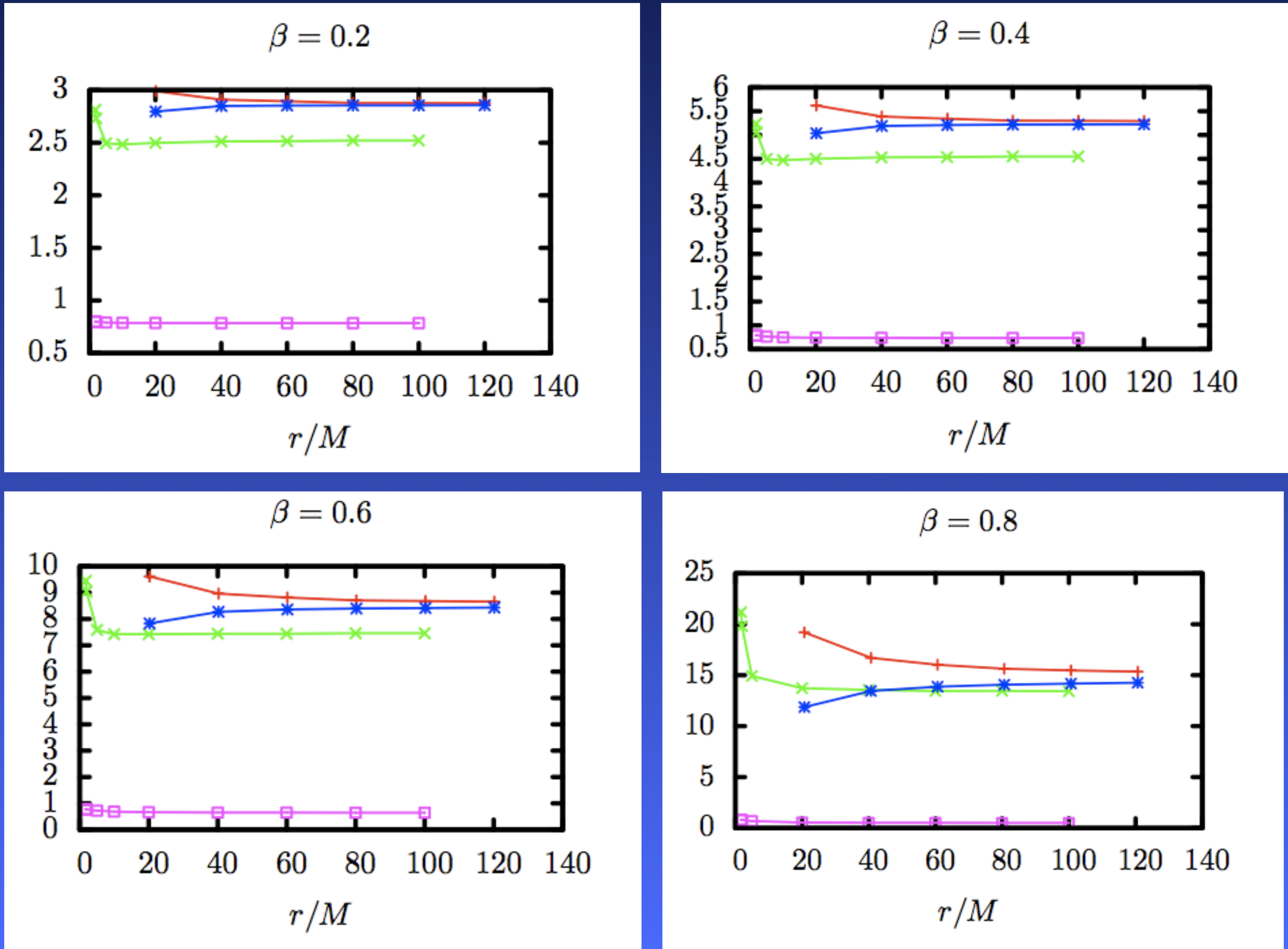
- Use an offset of 10M and boosts equal to 0.2, 0.4, 0.6 and 0.8.
- Construct two classes of surfaces: coordinate spheres and CE surfaces;
- Integrate two types of angular momentum over them:

$$\begin{aligned} J_{\text{Killing}} &= \frac{1}{8} \oint \phi^a R^b K_{ab} d^2 V \\ J_{\text{ADM}} &= \frac{1}{8} \oint \varphi^a R^b K_{ab} d^2 V \end{aligned}$$

Results



$\beta = 0$
1
0.95
0.9
0.85
0.8
0.75
0
20
40
60
80
100
120
140
r/M
$J_{\rm ADM}$, CS
$J_{\rm ADM}$, CE
$J_{\rm Killing}$, CS
$J_{\rm Killing}$, CE



β = 0.2
3
2.5
2
1.5
1
0.5
0 20 40 60 80 100 120 140
r/M
β = 0.4
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0 20 40 60 80 100 120 140
r/M
β = 0.6
10
9
8
7
6
5
4
3
2
1
0
0 20 40 60 80 100 120 140
r/M
β = 0.8
25
20
15
10
5
0
0 20 40 60 80 100 120 140
r/M

Discussion, future work

- Calculating angular momentum on coordinate spheres may lead to incorrect answers even in non-radiating spacetimes;
- CE surfaces seem to provide a better choice, as long as the rotational Killing vector field is also used in the angular momentum calculation.

- What is the exact relationship between the various prescriptions?
- Spacetimes with radiation.

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