XXI Conferenza SIGRAV
Università del Piemonte Orientale, Alessandria
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Quanti buchi neri può contenere l'universo?

Cosmologie analitiche e numeriche per un reticolo di singolarità di Schwarzschild

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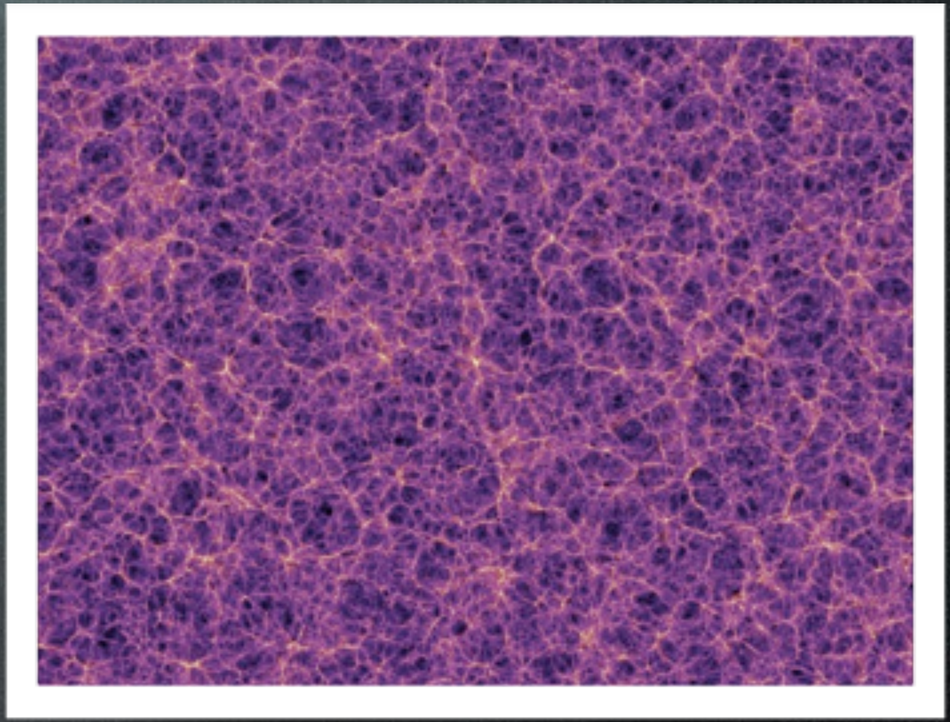
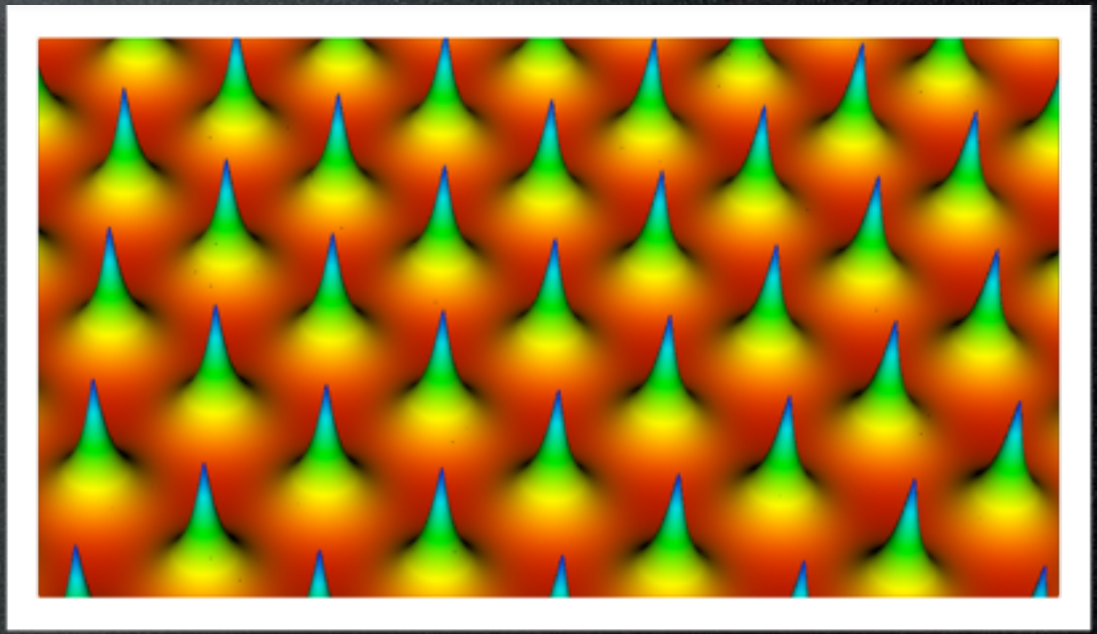
OVERVIEW

- **MOTIVATIONS**
  - Modelling cosmological inhomogeneities in a simple setting;
  - Studying the structure of initial data on periodic spaces;
- **HOW TO CONSTRUCT A PERIODIC BLACK-HOLE LATTICE**
  - S^3 lattices;
  - T^3 lattices;
- **ANALYSING A PERIODIC BLACK-HOLE LATTICE**
  - Length scaling
  - Observables
- **CONCLUSIONS**





MODELLING INHOMOGENEITIES

THE COSMOLOGICAL CONSTANT PROBLEM

Contemporary datasets probing cosmological distances cannot be accomodated by the Friedmann-Lemaître-Robertson-Walker class unless a cosmological constant is present.

Whilst the cosmological-constant term is perfectly plausible, its value is suspicious:

- it’s too small for vacuum energy density;
- it’s strangely close to the energy density due to matter.

Solutions:

- Quintessence;
- Higher-derivative gravity and other alternative theories;
- Quantum-inspired scenarios.

Modelling inhomogeneities

The fitting problem

Is this really new physics?

The fitting problem in cosmology (Ellis&Stoeger 1987): how should one map observations in a lumpy universe to exactly homogeneous and isotropic models, and what are the associated biases?

Global inhomogeneities:

- the cosmological principle is a reasonable, yet untestable ansatz.

Local inhomogeneities:

- modifications to global dynamics (the backreaction problem);
- modifications to optical properties.

Focus section on inhomogeneous cosmological models and averaging in cosmology in Classical and Quantum Gravity, August 2011.

Two main approaches: exact solutions and averaging schemes.

Modelling inhomogeneities

From small scale to large scale

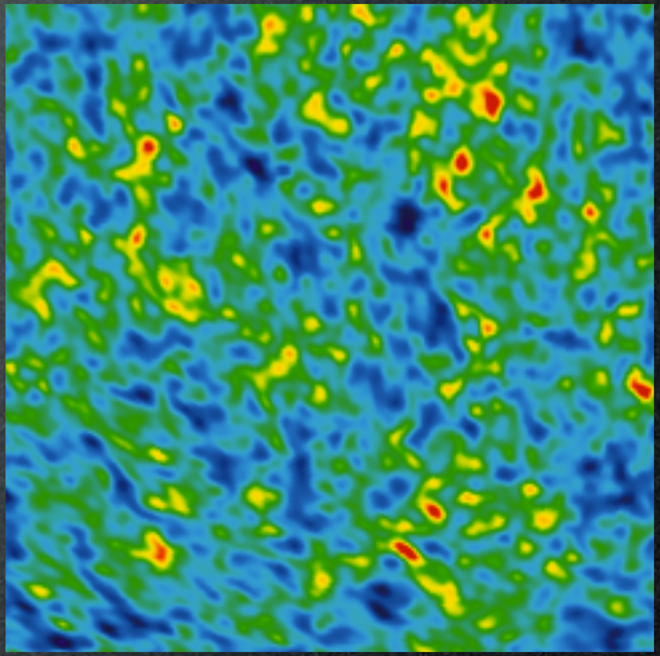
What is the metric tensor of the Universe?

Road I: how does one incorporate small-scale inhomogeneities in spaces that are homogeneous on larger scales?

Road II: how does one assemble a large-scale homogeneous space starting from inhomogeneous building blocks?

Modelling inhomogeneities

From small scale to large scale



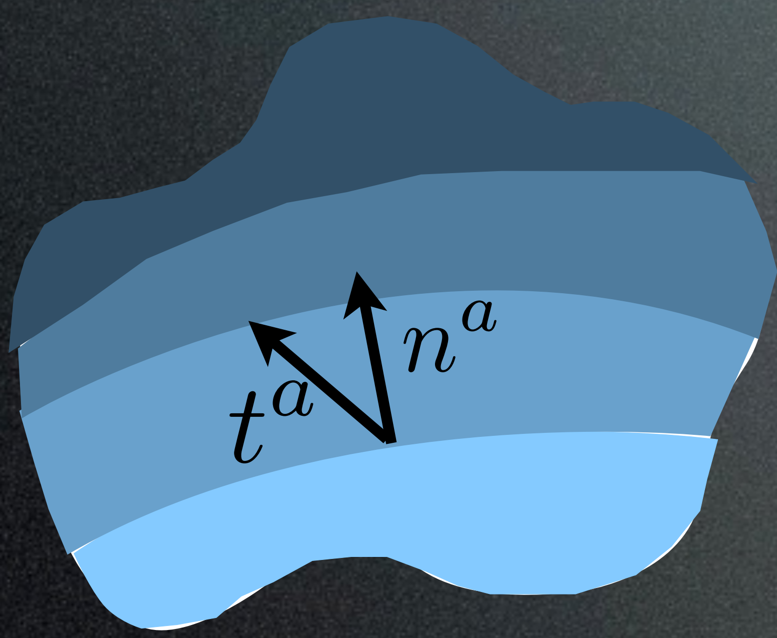
10 kpc — 10 Gpc

$s \ll D$ $s \gg D$

STRATEGY

THE 3+1 DECOMPOSITION

In order to formulate Einstein's equation as an initial-boundary value problem, one needs to choose a time coordinate and project the equations accordingly; reducing the system to first order form, one is left with twelve evolution four constraints equations:



$$t^a = \alpha n^a + \beta^a$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\gamma_{ij} = g_{ij} + n_i n_j$$

$$K_{ij} \equiv -\mathcal{L}_n \gamma_{ij}$$

$$\begin{aligned} R + K^2 - K_{ij}K^{ij} &= 0 \\ D_j K^j_i - D_i K &= 0 \end{aligned}$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^k_j + KK_{ij}) + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k$$

PERIODIC INITIAL DATA

THE EINSTEIN CONSTRAINTS WITH PERIODIC BOUNDARY CONDITIONS

What is the influence of periodic boundary conditions on the elliptic problem?

- No solutions in Newtonian's gravity!

Under which conditions do solutions exist? [Choquet-Bruhat, Isenberg & Pollack 2007]

- Integrability;

How much freedom is there to choose the physical data?

- Periodic boundary conditions are in a sense weaker than Dirichlet boundary conditions;
- Extra free data has to be provided;

What is the best numerical approach?

- Which algorithms are most natural to integrate with the extra conditions?
- How do the extra conditions affect the convergence of an algorithm [Elser, Rankenburg & Thibault 2006]?

Black-hole lattices

Constructing a black-hole lattice

Models studied initially by Lindquist&Wheeler (1957). Since then, several roads:

- Junction conditions [Clifton 2009]
- Series expansions [Bruneton&Larena 2012]
- Solving the constraints [Wheeler 1983, Clifton et al. 2012, Yoo et al. 2012, Bentivegna& Korzyński 2012, Yoo et al. 2013, Bentivegna&Korzyński 2013, Bentivegna 2014, Yoo&Okawa 2014]

$$\begin{aligned} R + K^2 - K_{ij}K^{ij} &= 0 \\ D_i K^i_j - D_j K &= 0 \end{aligned}$$

$$\begin{aligned} \gamma_{ij} &= \psi^4 \tilde{\gamma}_{ij} \\ K_{ij} &= \frac{K}{3}\gamma_{ij} + A_{ij} \end{aligned}$$

$$\begin{aligned} \tilde{\Delta}\psi - \tilde{R}\psi - \frac{K^2}{12}\psi^5 + \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij}\psi^{-7} &= 0 \\ \tilde{D}_i\tilde{A}^{ij} - \frac{2}{3}\psi^6\tilde{\gamma}^{ij}\tilde{D}_i K &= 0 \end{aligned}$$

Black-hole lattices

Constructing a black-hole lattice

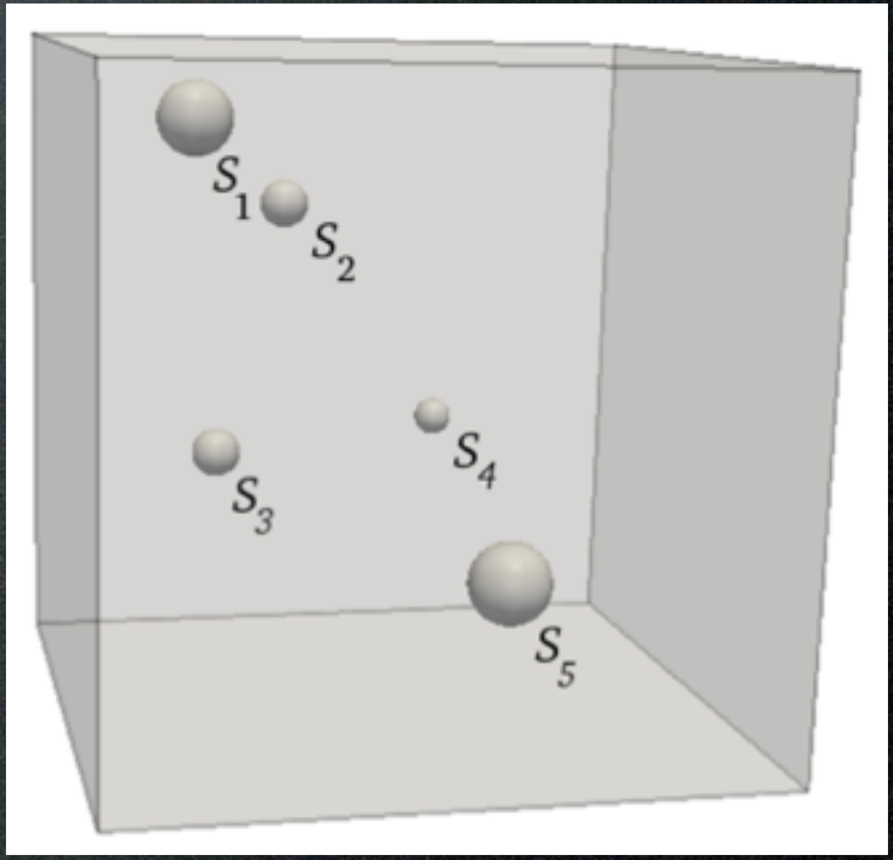
Further constraint:

$$\int_D \left(\frac{\tilde{R}}{8} \psi + \frac{1}{12} K^2 \psi^5 - \frac{1}{8} \psi^{-7} \tilde{A}^{ij} \tilde{A}_{ij} \right) \sqrt{\tilde{\gamma}}\, d^3x = 2\pi G \left(\int_D \rho \psi^5 \sqrt{\tilde{\gamma}}\, d^3x + \Sigma_{i=1}^{N} m_i \right)$$

On an asymptotically flat space, the surface terms at infinity and around the punctures cancel:

$$\psi = 1 + \frac{m}{2r}$$

However, there is no surface term on the periodic boundaries. In a periodic space, the extrinsic curvature and the scalar curvature cannot both be zero! No time symmetric, spatially-flat solution (homogeneous dust models have the same properties)



BLACK-HOLE LATTICES

CONSTRUCTING A BLACK-HOLE LATTICE

Two options:

- Keep a zero extrinsic curvature, but choose a conformal metric that is not flat [Wheeler 1983, Clifton et al. 2012]:

$$\tilde{\Delta}\,\psi - \frac{\tilde{R}}{8}\,\psi = 0$$

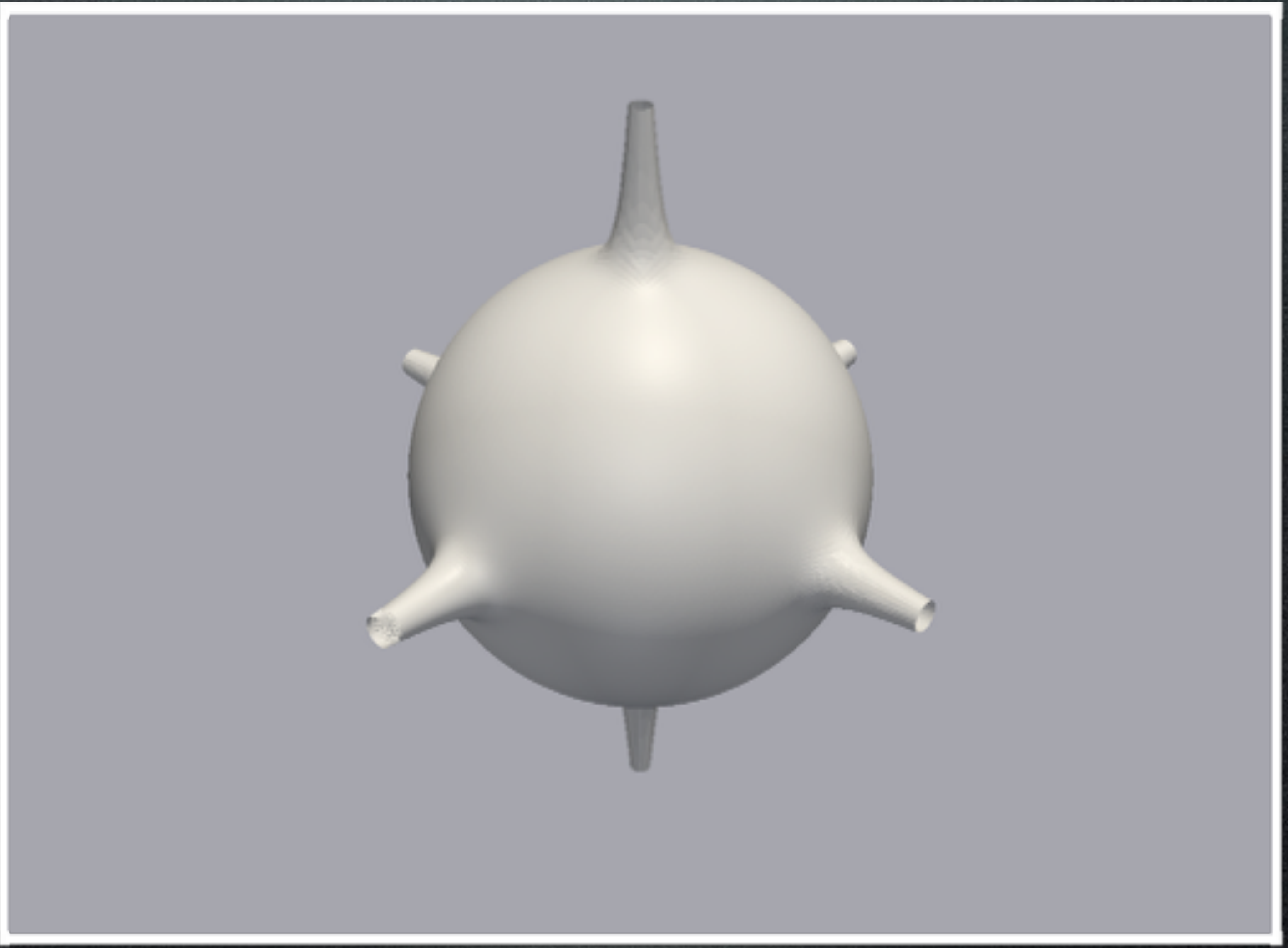
Notes:
1) Solutions only for positive scalar curvature (analogy to the FLRW class);
2) The hamiltonian constraint is linear! One can use the superposition principle to construct multi-black-hole solutions.

- Keep a flat conformal metric, but use a non-zero extrinsic curvature [Yoo et al. 2012]:

$$\Delta\psi - \frac{K^2}{12}\psi^5 = 0$$

Requires:
1) Numerical integration;
2) Extreme care with periodic boundaries.

K=0 Black-hole lattices



K=0 Black-hole lattices

Conformally-S^3 black-hole lattices

Hamiltonian constraint:

$$\tilde{\Delta}\,\psi - \frac{\tilde{R}}{8}\,\psi = 0$$

Spatial metric given by:

$$\mathrm{d}s^2 = \mathrm{d}\lambda^2 + \sin^2\lambda\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$$

$$\lambda \in [0,\pi]$$
$$\theta \in [0,\pi]$$
$$\varphi \in [0,2\pi]$$

A solution:

$$\phi(\lambda) = \frac{A}{\sin\lambda/2}$$

K=0 Black-hole lattices

Conformally-S3 black-hole lattices

Multiple black holes can be obtained by superimposing this fundamental solution. It is convenient to embed this three-sphere in R4, and to express the solution in this coordinate space:

$$\phi(\bar{X}) = \sum_{i=1}^{N} \frac{A_i}{\sin \lambda_i / 2} = \sum_i A_i \sqrt{\frac{2}{1 - \bar{X} \cdot \bar{N}_i}}$$

The parameters A_i and the black-hole centers are arbitrary, but if one is interested in regular lattices these have to be chosen carefully. In particular, the parameters A_i have to be the same, and the centers have to be equidistant from each other.

Curious cases:

$N = 1$

$$\mathrm{d}s^2 = \frac{1}{\sin^4 \frac{\lambda}{2}} \left(\mathrm{d}\lambda^2 + \sin^2 \lambda \left(\mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\varphi^2\right)\right)$$

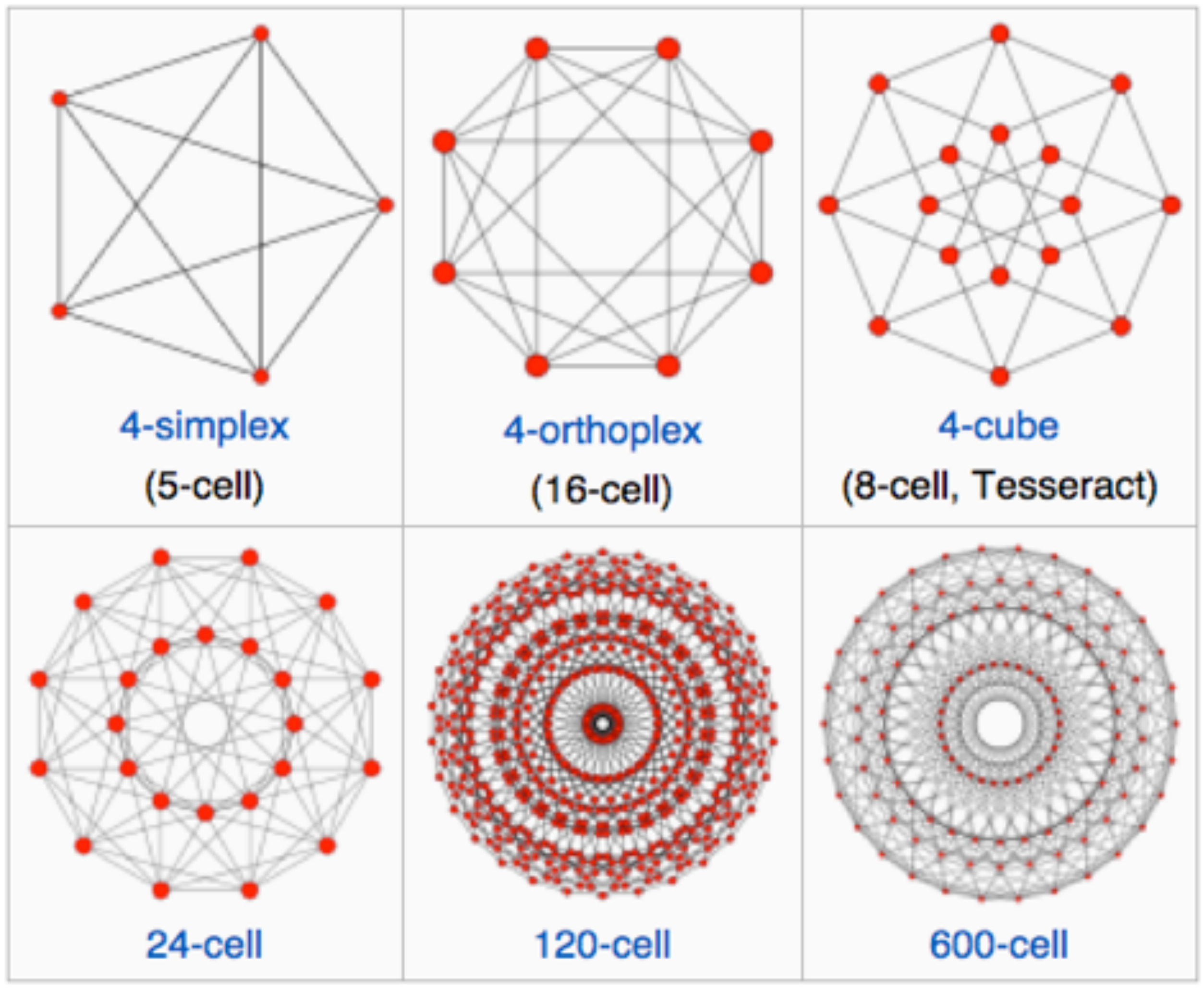
$N = 2$

$$\mathrm{d}s^2 = \left(\frac{1}{\sin \frac{\lambda}{2}} + \frac{1}{\sin \frac{\lambda - \pi}{2}}\right)^4 \left(\mathrm{d}\lambda^2 + \sin^2 \lambda \left(\mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\varphi^2\right)\right)$$

$K=0$ Black-hole lattices

The regular tessellations of S^3

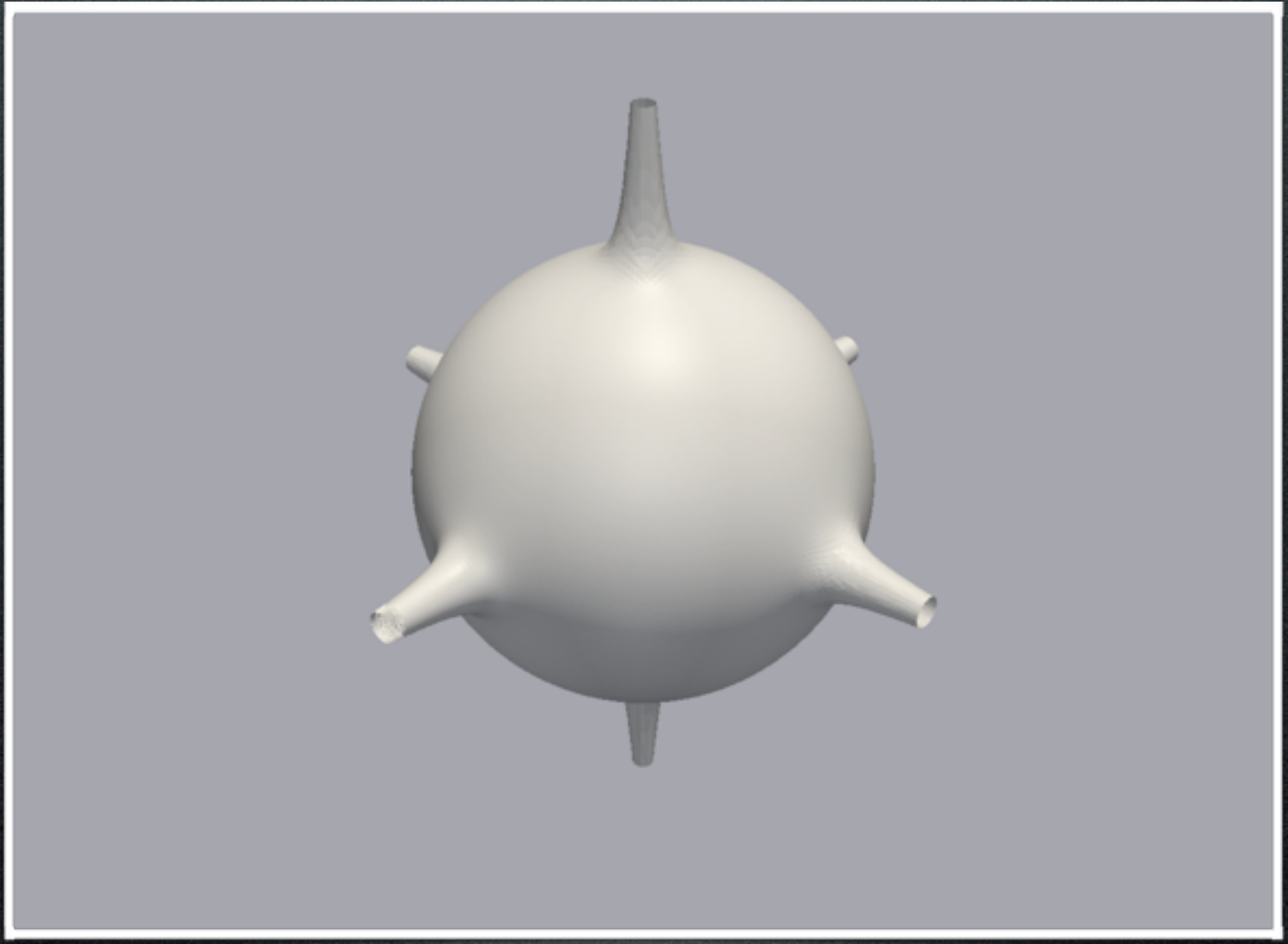
On a three-sphere, there is only a finite number of “regular” arrangements of points, corresponding to the regular tessellations of S^3. N can be equal to 5, 8, 16, 24, 120, 600.



en.wikipedia.org

K=0 Black-hole lattices

The regular tessellations of S^3

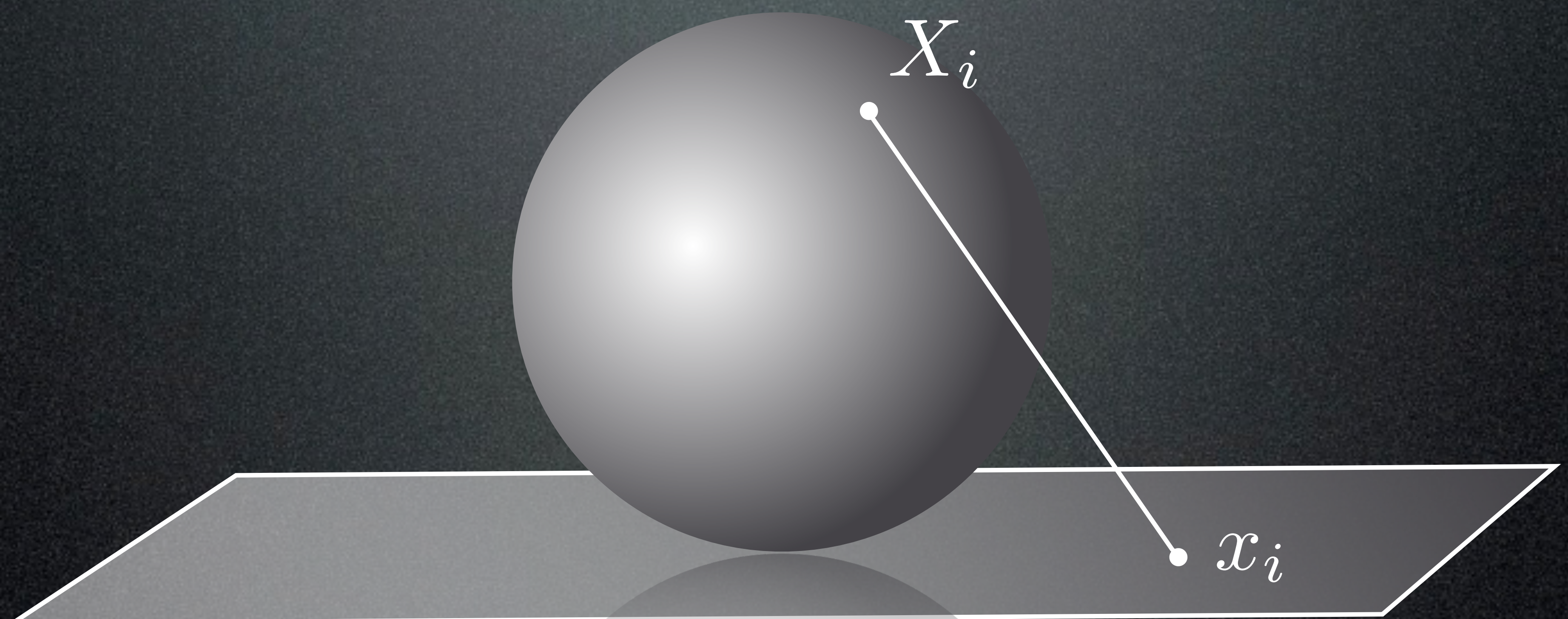


K=0 Black-hole lattices

A stereographic projection from S^3 onto R^3

Coordinate transformation to simplify numerical treatment:

$$(X^1, X^2, X^3, X^4) \to (x^1, x^2, x^3) : x^i = \frac{2X^i}{1 - X^4}$$

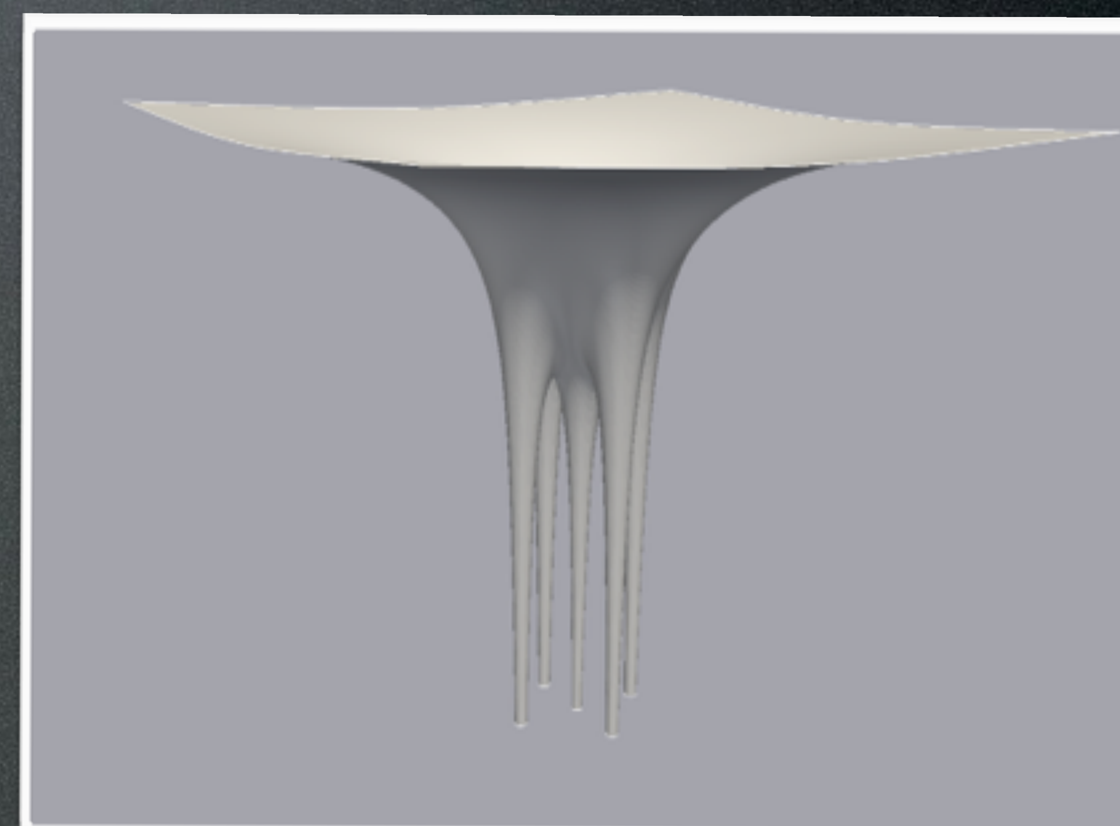


K=0 Black-hole lattices

A stereographic projection from S^3 onto R^3

$$\tilde{\psi}(\vec{y}) = 1 + \sum_{i=2}^{N} \frac{m_i}{2\left|\vec{y} - \vec{\mathcal{N}}_i\right|}$$

Brill-Lindquist initial data!



K=0 Black-hole lattices

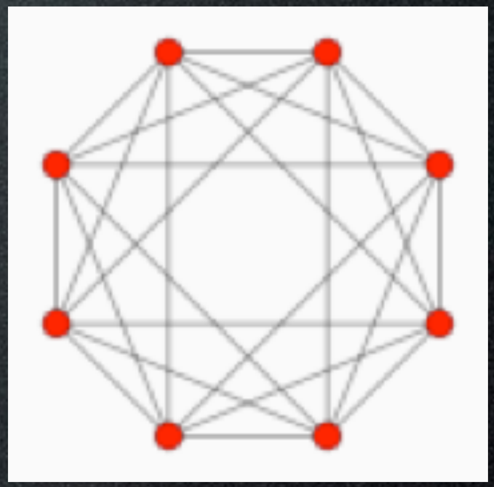
The 8-black-hole universe

We have studied the 8-black-hole case and its full-GR evolution:

$$
\begin{aligned}
\bar{N}_1 &= (1,0,0,0)\,, \\
\bar{N}_2 &= (-1,0,0,0)\,, \\
\bar{N}_3 &= (0,1,0,0), \\
\bar{N}_4 &= (0,-1,0,0), \\
\bar{N}_5 &= (0,0,1,0)\,, \\
\bar{N}_6 &= (0,0,-1,0)\,, \\
\bar{N}_7 &= (0,0,0,1), \\
\bar{N}_8 &= (0,0,0,-1).
\end{aligned}
$$

→

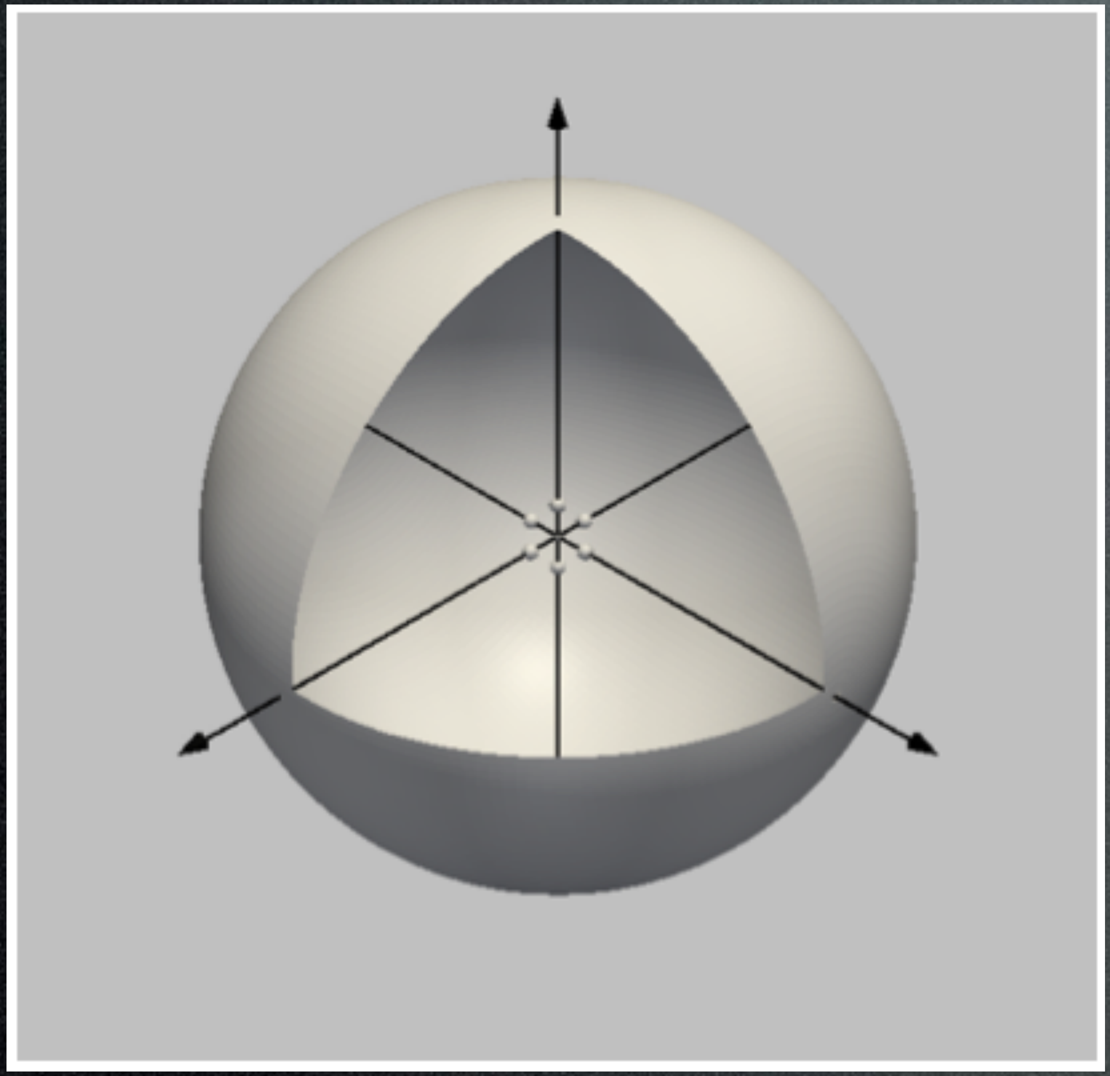
$$
\begin{aligned}
\vec{\mathcal{N}}_2 &= (0,0,0)\,, \\
\vec{\mathcal{N}}_3 &= (2,0,0), \\
\vec{\mathcal{N}}_4 &= (-2,0,0), \\
\vec{\mathcal{N}}_5 &= (0,2,0)\,, \\
\vec{\mathcal{N}}_6 &= (0,-2,0)\,, \\
\vec{\mathcal{N}}_7 &= (0,0,2), \\
\vec{\mathcal{N}}_8 &= (0,0,-2).
\end{aligned}
$$

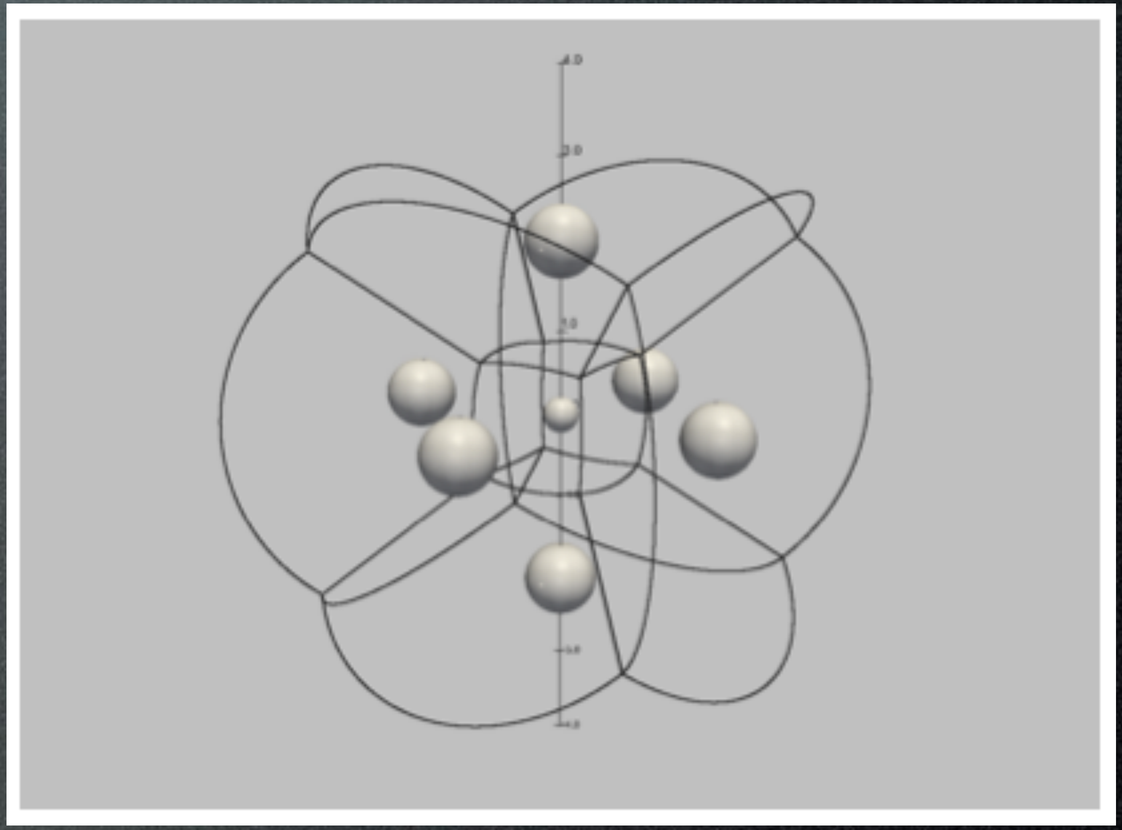


K=0 Black-hole lattices

The 8-black-hole universe

Initial horizons (the outermost surface is inner trapped!):





K=0 Black-hole lattices

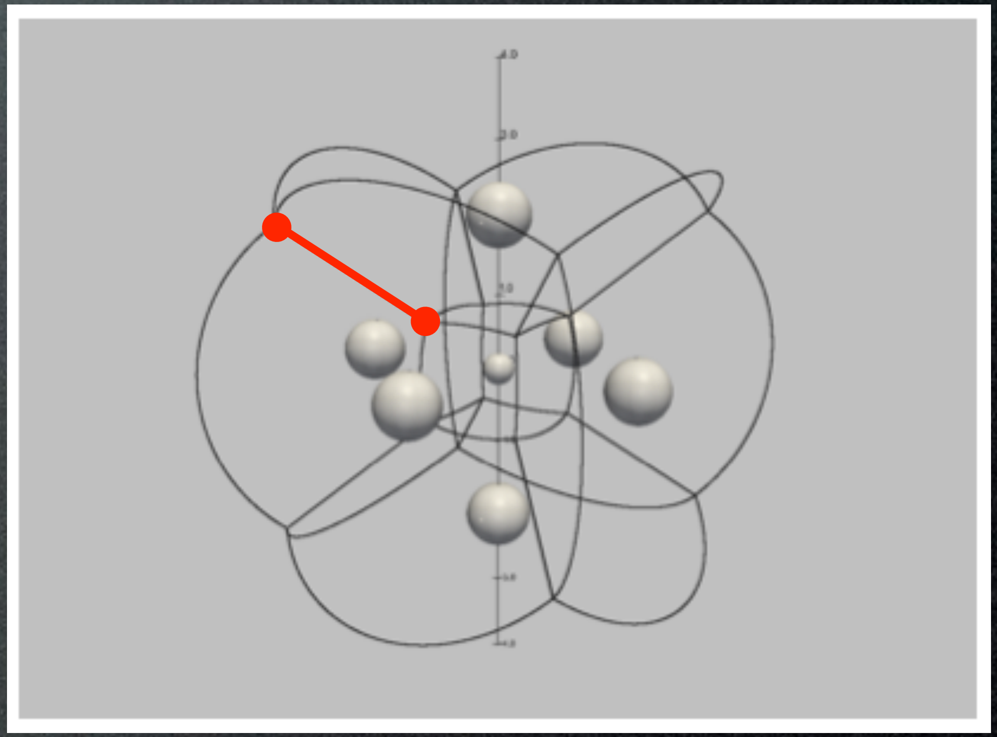
The 8-black-hole universe

For a fair comparison with the FLRW class, one needs to elect some measure of (proper) distance, and measure its scaling in proper time.

$$\tau(t,x) = \int_0^t \alpha(t',x)\mathrm{d}t'$$

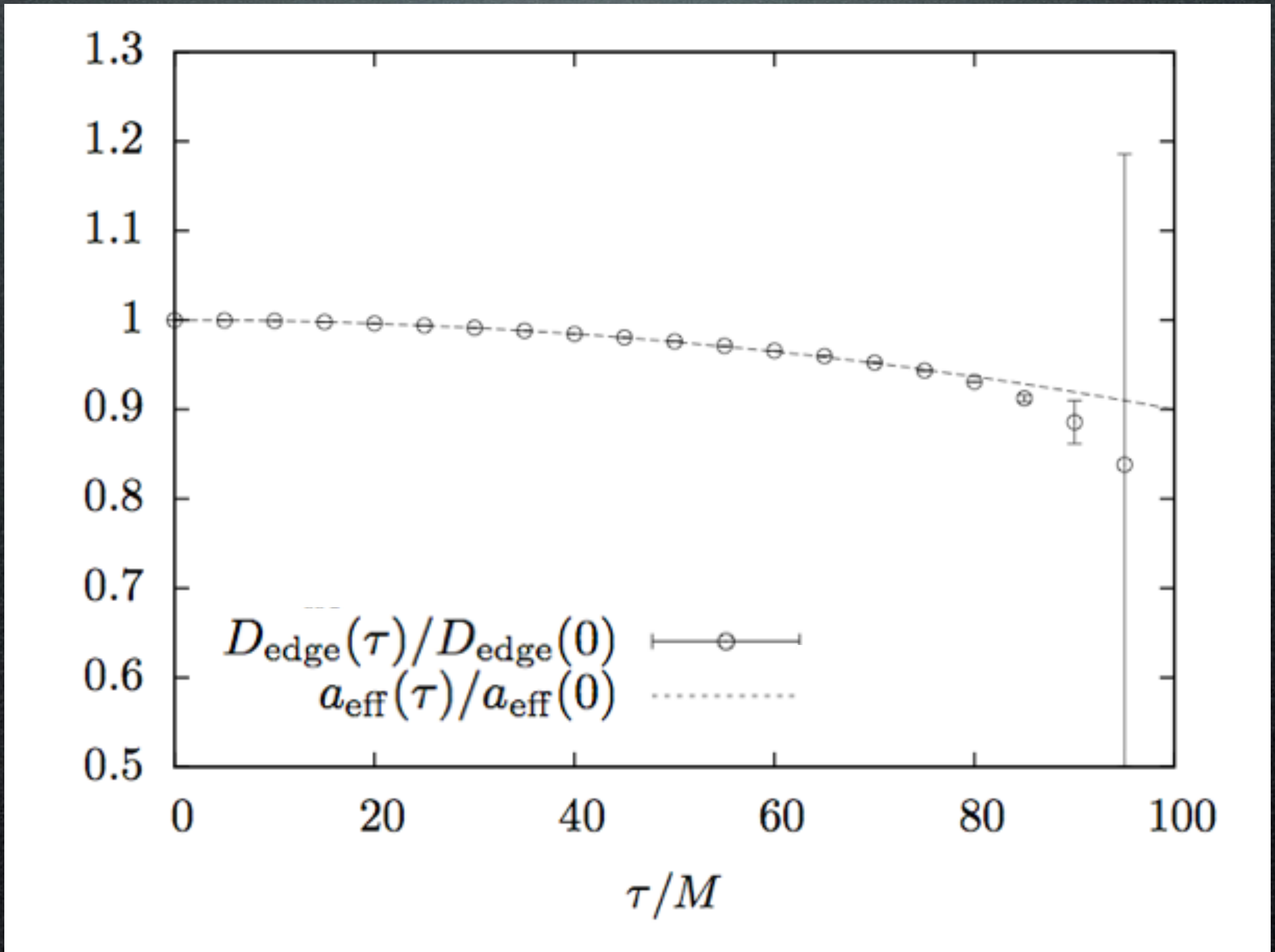
$$x_{\mathrm{g}}(t, x_{\mathrm{init}}) = x_{\mathrm{g}}(t-\Delta t, x_{\mathrm{init}}) - \int_{t-\Delta t}^{t} \beta^x(t', x_{\mathrm{g}}(t-\Delta t, x_{\mathrm{init}}))\mathrm{d}t'$$

$$D(\tau) = \int_{\gamma_\tau} \left[(-\alpha^2(\tau,\ell) + \beta^2(\tau,\ell))(\partial_\ell t)^2 + \beta_i(\tau,\ell)\partial_\ell t \partial_\ell x^i + \gamma_{ij}(\tau,\ell)\partial_\ell x^i \partial_\ell x^j\right]^{1/2} \mathrm{d}\ell$$



K=0 BLACK-HOLE LATTICES

THE 8-BLACK-HOLE UNIVERSE



K=0 Black-hole lattices

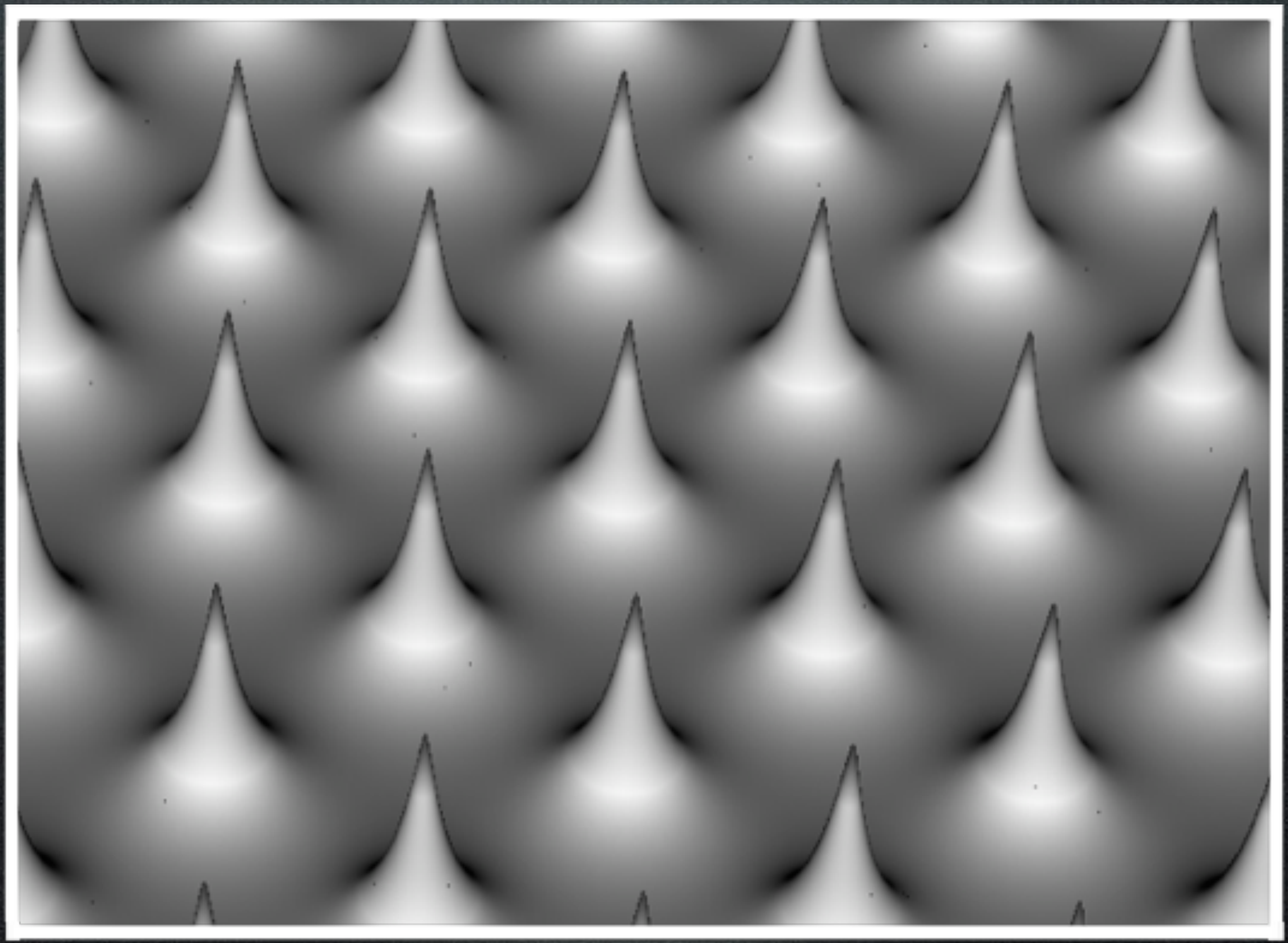
The 8-black-hole universe

- The large-scale behavior of the system is just that of a homogeneous space filled with dust. (Even if the inhomogeneities are relatively large-scale!)
- The density parameter, however, appears dressed by the inhomogeneities.

$$M_{\mathrm{eff}} = \rho_{\mathrm{eff}}\, 2\pi^2 a_{\mathrm{eff}}^3 = 378.78, \quad M_{8\mathrm{BH}} = 8M_{\mathrm{ADM}} = 303.53$$

- In this case, the conformal-data part of the ID plays a rather decisive role in its evolution. Is this always the case?

R=0 Black-hole lattices



R=0 Black-hole lattices

Conformally-T^3 black-hole lattices

Hamiltonian constraint:

$$\Delta\psi - \frac{K^2}{12}\psi^5 + \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij}\psi^{-7} = 0$$

It requires numerical integration. If K is not a spatial constant or $\tilde{A}_{ij}$ is not transverse, the momentum constraint has to be solved as well. In all cases, the solution has to include a mechanism to preserve the integrability condition (see Part IV).

In this case, the constraint takes the form:

$$\int_{\text{cell}} \left(\frac{K^2}{12}\psi^5 + \frac{1}{8}\tilde{A}_{ij}A^{ij}\psi^{-7} \right) = 0$$

This has to be enforced iteratively since it depends on the unknown conformal factor (and potentially on the extrinsic curvature). If this condition is not satisfied, the system does not admit solutions ("singular")! The extent to which one can reduce the equation residual depends strongly on how well we can satisfy the compatibility condition.

$R=0$ Black-hole lattices

Conformally-T^3 black-hole lattices

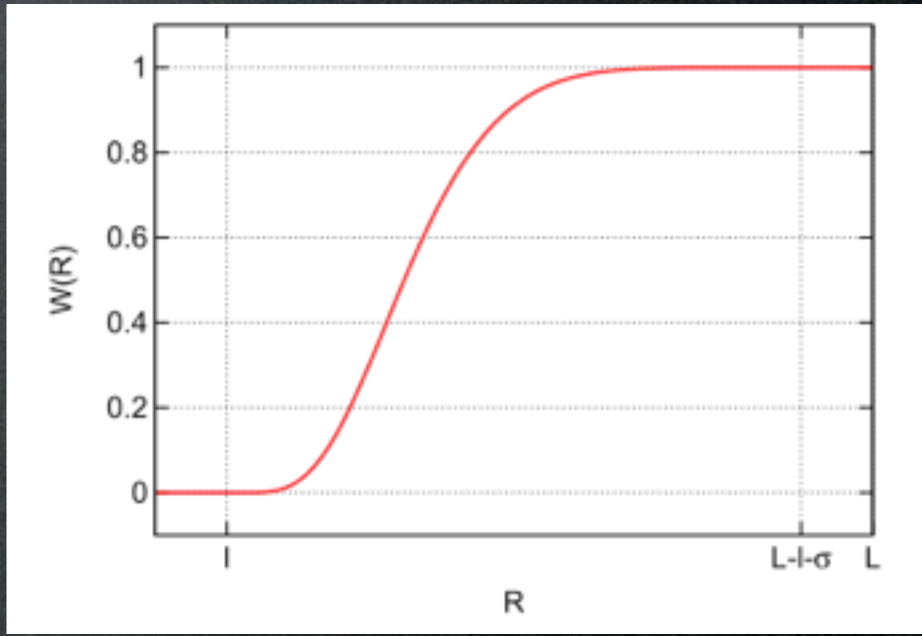
A prescription: Yoo et al. (2012) construct an initial-data slice that is asymptotically Schwarzschild (in the static slicing) next to the center, and asymptotically CMC (with a negative mean curvature) next to the cell faces.

$$K(\boldsymbol{x}) = -3H_{\rm eff}W(R),$$

$$\psi(\boldsymbol{x}) := \Psi(\boldsymbol{x}) - \frac{M}{2R}[1 - W(R)].$$

$$\triangle\psi = \triangle\left(\frac{M}{2R}W(R)\right) - \frac{1}{8}(\tilde{L}X)_{ij}(\tilde{L}X)^{ij}\Psi^{-7} + \frac{1}{12}K^2\Psi^5,$$

$$\triangle Z = \frac{1}{2}\partial_i(\Psi^6\partial^i K), \qquad \triangle X^i = -\frac{1}{3}\partial^i Z + \frac{2}{3}\Psi^6\partial^i K.$$



R=0 Black-hole lattices

Conformally-T^3 black-hole lattices

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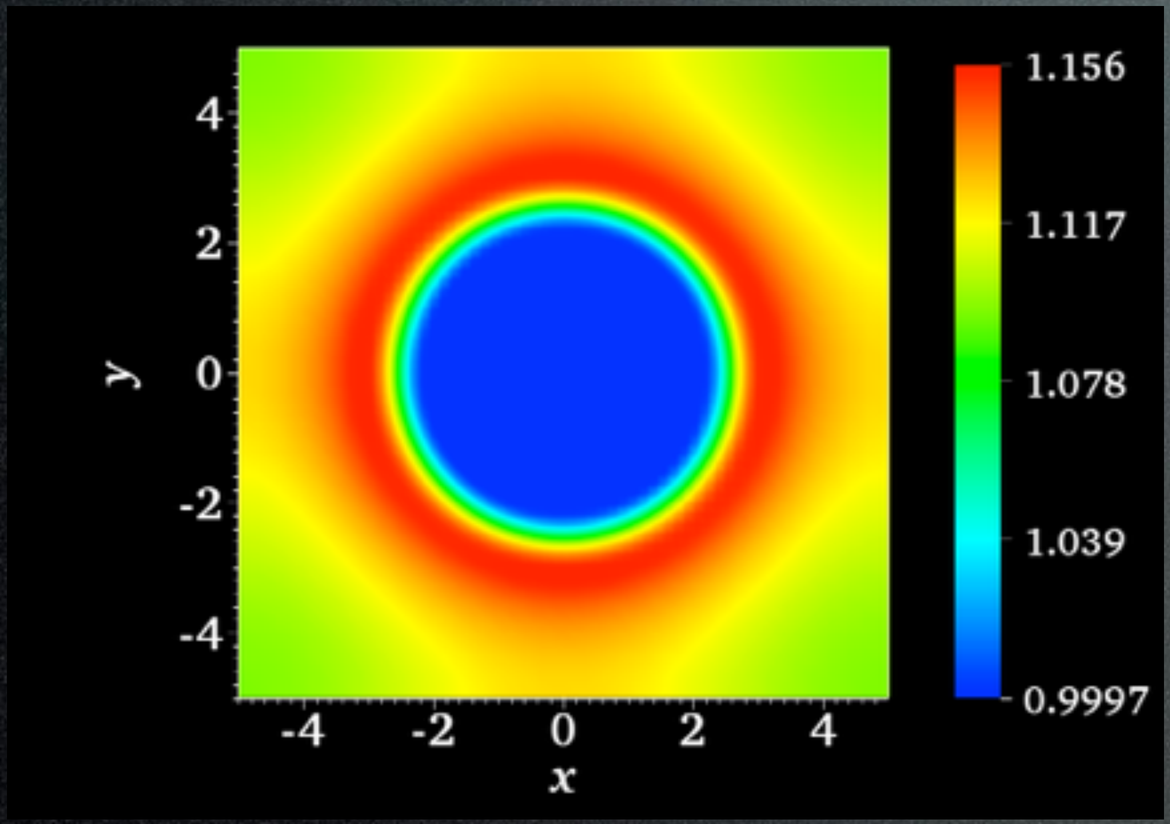
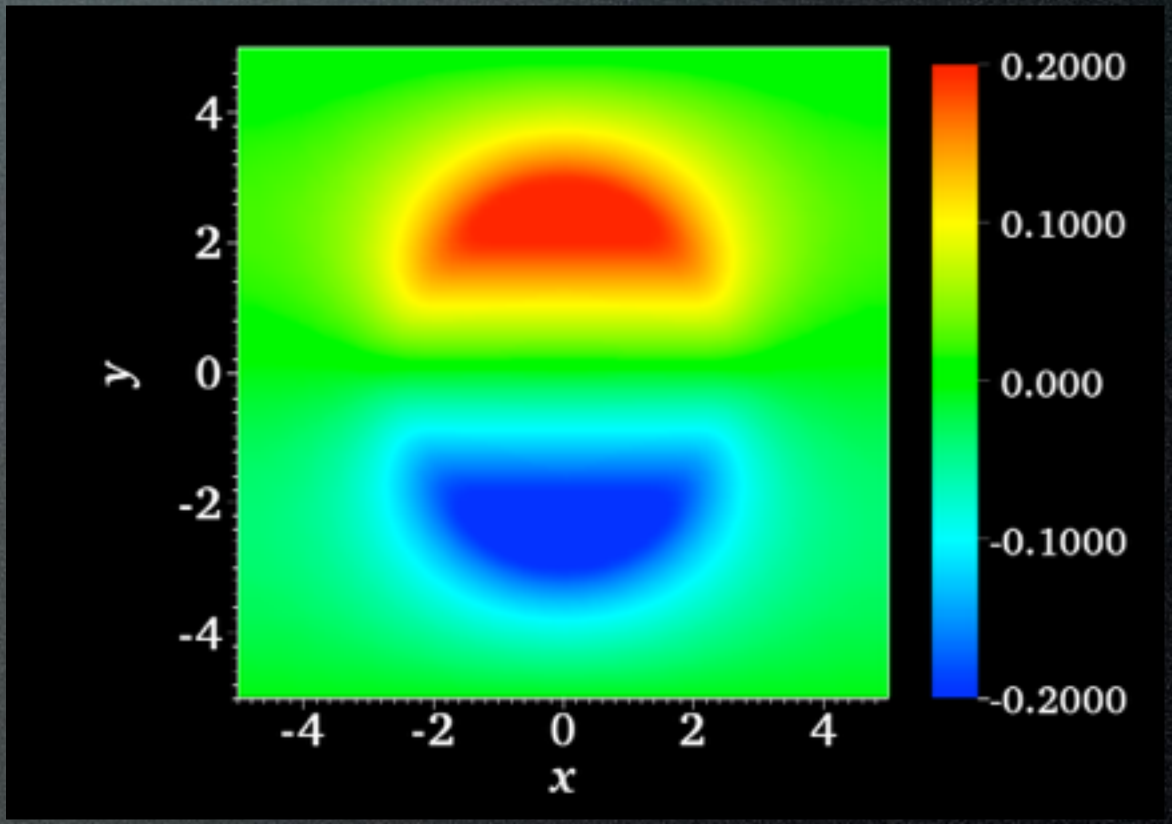
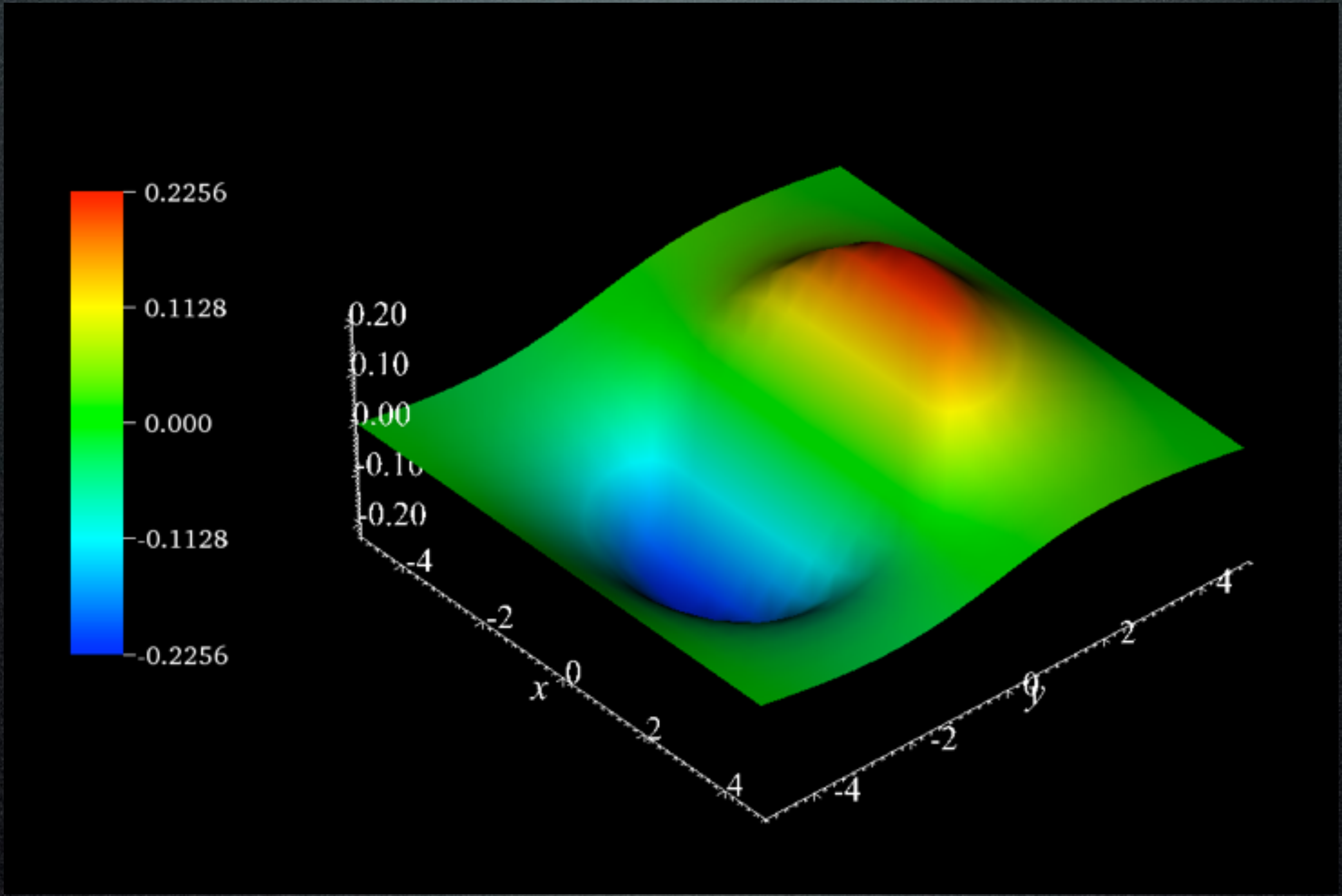


X^x



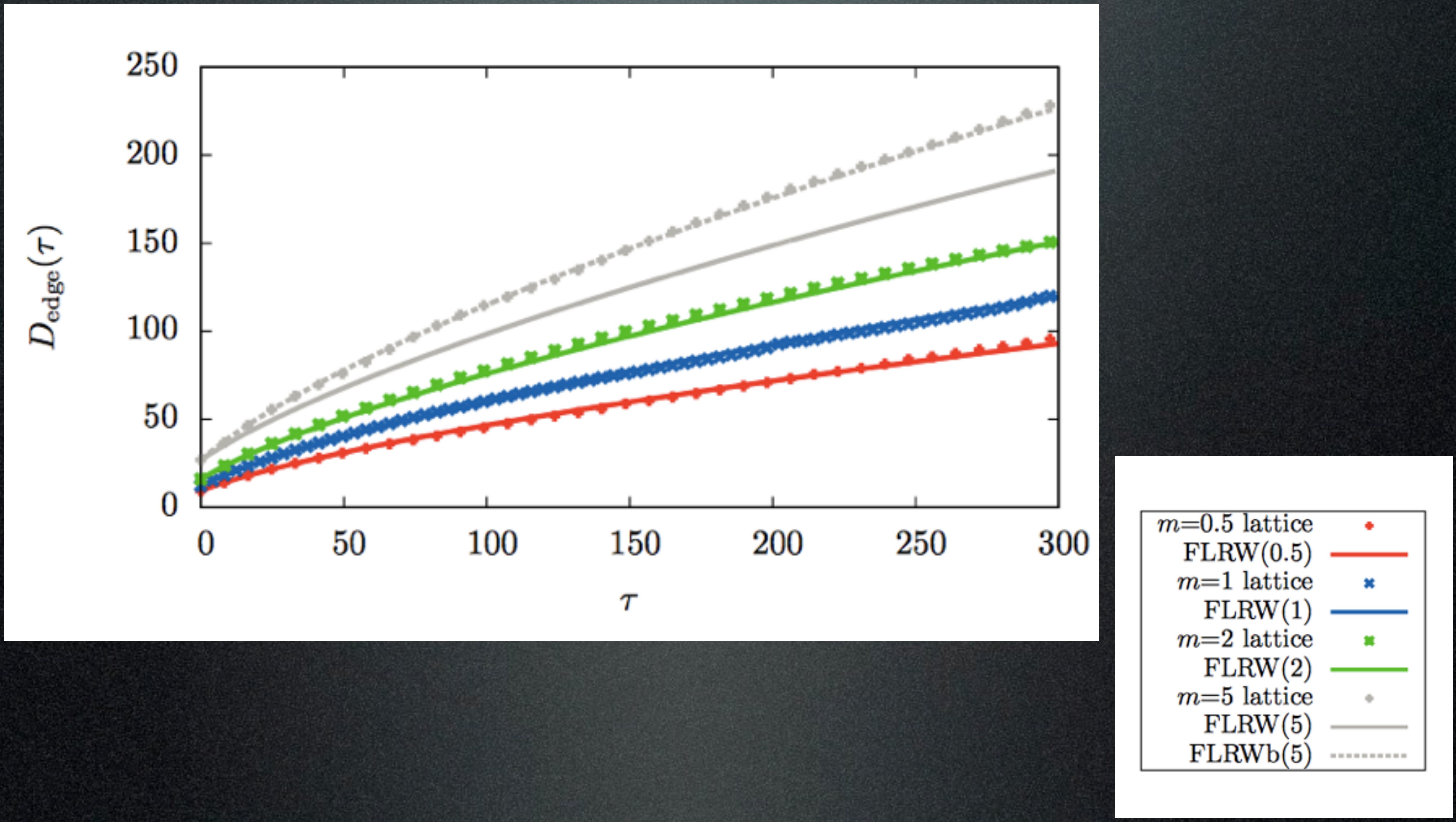
R=0 Black-hole lattices

Conformally-T^3 black-hole lattices



R=0 Black-hole lattices

Conformally-T³ black-hole lattices



Conclusions

- Building blocks cannot be assembled arbitrarily.
- The conditions for the existence of a periodic black-hole lattice reproduce some of the features of the FLRW class (in particular, the time-symmetry *vs.* topology relationship). Their kinematical behavior is also remains close to the FLRW counterpart.
- Is this the full story?
- This is just one of the current applications of Numerical Relativity. Check out "Exploring New Physics Frontiers Through Numerical Relativity", upcoming Living Review in Relativity, arXiv:1409.0014.